October Math Gems

Problem of the week 12

§1 Problems

Problem 1.1. If a, b are real numbers such that $a \geq b$, prove that

$$\left(\frac{\sqrt{b}(\sqrt{a}+\sqrt{b})}{8b} \right)^2 \geq \frac{1}{16}.$$

Solution. We have to prove that

$$\left(\frac{\sqrt{b}(\sqrt{a}+\sqrt{b})}{8b} \right)^2 \geq \frac{1}{4}.$$

First, observe that

$$\left(\frac{\sqrt{b}(\sqrt{a}+\sqrt{b})}{8b} \right)^2 = b\left(\frac{\sqrt{a}}{8b} + \frac{\sqrt{b}}{8b} \right)^2 = b(\frac{a}{64b^2} + \frac{b}{64b^2} + \frac{2\sqrt{ab}}{64b^2})$$

Since $a \geq b$, we have that

$$b(\frac{a}{64b^2} + \frac{b}{64b^2} + \frac{2\sqrt{ab}}{64b^2}) \geq \frac{b}{64b} + \frac{b}{64b} + \frac{2\sqrt{b^2}}{64b} = \frac{4}{64} = \frac{1}{16}$$

□

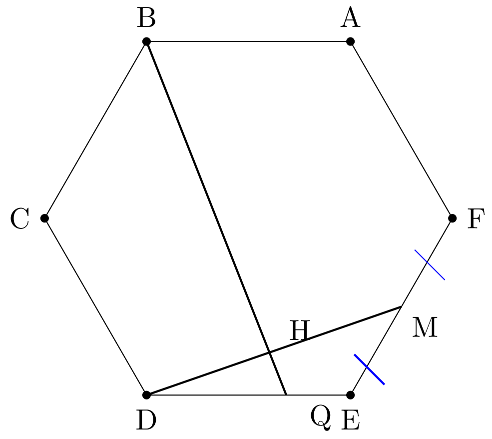
Problem 1.2. We have 6 balls of different colours, including blue and red. In how many ways can we arrange them in a row so that the blue ball and red ball do not come together.

Solution. Imagine that we first place the other 4 balls, then choose places for the red and blue balls. Hence, the number of ways to arrange 4 balls is ${}_4P_4 = 4!/0! = 24$ ways.

Now, putting these two balls in two different places among the 5 places available (3 between the balls, 2 at the edges) satisfies the condition of separating them. Hence, the number of ways of separating them $= {}_5P_3 = 5!/3! = 20$ ways.

Hence, the total number of ways $= 20 \cdot 24 = 480$ ways. □

Problem 1.3. ABCDEF is a regular hexagon. Given that M is the midpoint of $\overline{\text{EF}}$ and $\overline{\text{BQ}}$ is perpendicular to $\overline{\text{DM}}$ at H. Find the ratio $\frac{DH}{HM}$.



Solution. Assume ABCDEF is a unit regular hexagon. Hence, using cosine law for triangle DCB, with $DC = CB = 1$ and $m(\angle BCD) = 2\pi/3$, we get $DB = \sqrt{3}$. Using it for triangle BFM with $BF = \sqrt{3}, MF = 1/2$ and $m(\angle MFB) = \pi/2$, we get $MB = \sqrt{13}/2$. Using it for triangle DEM with $DE = 1, ME = 1/2$ and $m(\angle DEM) = 2\pi/3$, we get $DM = \sqrt{7}/2$. Since triangles BDH, BMH are right, we apply Pythagoras' theorem to obtain

$$\begin{cases} 3 - (DH)^2 = (BH)^2 \\ 13/4 - (\sqrt{7}/2 - DH)^2 = (BH)^2 \end{cases}$$

Solving for DH, we get $DH = \frac{3\sqrt{7}}{14}$. Hence, $HM = \frac{\sqrt{7}}{2} - \frac{3\sqrt{7}}{14} = \frac{4\sqrt{7}}{14}$.
Thus, $\frac{DH}{HM} = \frac{3}{4}$ □

Problem 1.4. Find all pairs of integers (a, b) such that

$$a^2 + b = 2023.$$

Solution. The given condition is equivalent to $a^2 = b(b^{2022} - 1)$. If $b \geq 2$, then b and $b^{2022} - 1$ are both positive and relatively prime, hence both perfect squares. However, since b^{2022} is also a perfect square, we get a contradiction. Hence, $b \leq 1$. Moreover, $b \leq -2$ is impossible because it implies that $a^2 < 0$. Hence, the possible values for b are -1,0,1. All of which give $a = 0$ as the only valid solution.
Hence, the pairs are $(0, -1), (0, 0), (0, 1)$. □

Problem 1.5. Equal letters stand for equal numbers, different letters for different numbers.

$$4 \cdot ABCDE = EDCBA.$$

Determine $ABCDE$.

Solution. Since $EDCBA$ is a 5-digit number, we know that $ABCDE < 1/4 \cdot 100,000 = 25000$. Thus, $A = 1$ or 2. But since $EDCBA$ is a multiple of 4, it is even. Thus, $A = 2$. Moreover, $4 \cdot XXXXE = XXXX2$. Hence, $E = 3$ or 8. However, $2XXXX \cdot 4 \neq 3XXXX$. Thus, $E = 8$.
We have $2BCD8 \cdot 4 = 8DCB2$. Hence, $4 \cdot BCD + 3 = DCB$. We have that $B < 2$ since it can not equal 2 and there would be a carry for values greater than 2. Hence, $B = 1$. Trying different cases for C and D, we get that $ABCDE = 21978$ □

Problem 1.6. Find all integers satisfying the equation

$$2^x \cdot (4-x) = 2x+4$$

Solution. Since 2^x must be positive, we have $\frac{2x+4}{4-x} > 0$ yielding $-2 < x < 4$. Thus, it suffices to check the points -1, 0, 1, 2, 3. The three solutions are $x = 0$, 1,2. $\square$

Problem 1.7. $a_1a_2a_3$ and $a_3a_2a_1$ are two three-digit decimal numbers, with a_1, a_3 being different non-zero digits. The squares of these numbers are five-digit numbers $b_1b_2b_3b_4b_5$ and $b_5b_4b_3b_2b_1$ respectively. Find all such three-digit numbers.

Solution. Assume $a_1 > a_3 > 0$. As the square of $a_1a_2a_3$ must be a five-digit number we have $a_1 \leq 3$. Now a straightforward case study shows that $a_1a_2a_3$ can be 301, 311, 201, 211 or 221. $\square$

Problem 1.8. Let's call a positive integer "interesting" if it is a product of two (distinct or equal) prime numbers. What is the greatest number of consecutive positive integers all of which are "interesting"?

Solution. The three consecutive numbers $33 = 3(11)$, $34 = 2(17)$ and $35 = 5(7)$ are all "interesting". On the other hand, among any four consecutive numbers there is one of the form $4k$ which is "interesting" only if $k = 1$. But then we have either 3 or 5 among the four numbers, neither of which is "interesting". $\square$

Problem 1.9. Solve the system of equations:

$$\begin{cases} x^5 = y + y^5 \\ y^5 = z + z^5 \\ z^5 = t + t^5 \\ t^5 = x + x^5 \end{cases}$$

Solution. Adding all four equations we get $x + y + z + t = 0$. On the other hand, the numbers x, y, z, t are simultaneously positive, negative or equal to zero. Thus, $x = y = z = t = 0$ is the only solution. $\square$

Problem 1.10. Determine all real numbers a, b, c, d that satisfy the following system of equations.

$$\begin{cases} abc + ab + bc + ca + a + b + c = 1 \\ bcd + bc + cd + db + b + c + d = 9 \\ cda + cd + da + ac + c + d + a = 9 \\ dab + da + ab + bd + d + a + b = 9 \end{cases}$$

Solution. Substituting $A = a+1, B = b+1, C = c+1, D = d+1$, we obtain

$$ABC = 2 \tag{1}$$
$$BCD = 10 \tag{2}$$
$$CDA = 10 \tag{3}$$
$$DAB = 10 \tag{4}$$

Multiplying (1), (2), (3) gives $C^3(ABD)^2 = 200$, which together with (4) implies C^3 $= 2$. Similarly we find $A^3 = B^3 = 2$ and $D^3 = 250$. Therefore,the only solution is $a = b = c = \sqrt[3]{2} - 1$ and $d = 5\sqrt[3]{2} - 1$ $\square$

Problem 1.11. Find all pairs of integers (p, q) such that

$$(p-q)^2 = \frac{4pq}{p+q-1}$$

Solution. By multiplying both sides of the equation with the denominator of the right side, we get $(p-q)^2(p+q-1) = 4pq$, which gives $(p+q)^2 = (p-q)^2(p+q)$. Hence, $p+q=0$ or $p+q=(p-q)^2$. The first case gives $p=-q$, i.e. all the pairs $(n,-n)$ where n is integer, are suitable. In the second case take $p-q=n$, then $p+q=n^2$ and $p=\frac{n^2+n}{2}, q=\frac{n^2-n}{2}$.
To ensure the denominator of the fraction in the problem is not zero, the condition $p+q \neq 1$ must be added. Hence, $n \neq 1, n \neq -1$.
Pairs $(n,-n), \left(\frac{k(k+1)}{2}, \frac{k(k-1)}{2}\right)$ for an arbitrary integer n, excluding $(1,0),(0,1)$. ☐

Problem 1.12. Assume that $g(x) = \frac{x^2}{1+x^2})$. Find the value of the expression

$$g(\frac{1}{2000}) + g(\frac{2}{2000}) + \cdots + g(\frac{1999}{2000}) + g(\frac{2000}{2000}) + g(\frac{2000}{1999}) + \cdots + g(\frac{2000}{1})$$

Solution. 1999.5. One obtains the answer using the fact that for any non-zero real number x,

$$\begin{aligned} f(x) + f(\frac{1}{x}) &= \frac{x^2}{1+x^2} + \frac{(\frac{1}{x})^2}{1+(\frac{1}{x})^2} = \frac{x^2}{1+x^2} + \frac{1}{x^2} \cdot \frac{1}{1+\frac{1}{x^2}} \\ &= \frac{x^2}{1+x^2} + \frac{1}{1+x^2} = 1. \end{aligned}$$

☐

Problem 1.13. Let a, b, c and d be non-negative integers. Prove that the numbers 2^a7^b and 2^c7^d give the same remainder when divided by 15 if and only if the numbers $3^a5^b$and 3^c5^d give the same remainder when divided by 16.

Solution. First, we show that if $|a'-a| = |b'-b| = 2$, then $2^a7^b \equiv 2^{a'}7^{b'} \pmod{15}$ and $3^a5^b \equiv 3^{a'}5^{b'} \pmod{16}$. Indeed, we can assume that $a' = a+2$. If $b' = b+2$, we obtain

$$2^{a'}7^{b'} = 2^a7^b \cdot 2^27^2 = 2^a7^b \cdot (2 \cdot 7)^2 \equiv 2^a7^b \cdot (-1)^2 = 2^a7^b \pmod{15}.$$

Moreover,

$$3^{a'}5^{b'} = 3^a5^b \cdot 3^25^2 = 3^a5^b \cdot (3 \cdot 5)^2 \equiv 3^a5^b \cdot (-1)^2 = 3^a5^b \pmod{16}.$$

If $b' = b-2$, we can use the same relations noting that $7^4 \equiv 1 \pmod{15}$ and $5^4 \equiv 1 \pmod{16}$. Now we prove that for every pair of non-negative integers (a,b) there exists a pair (a',b') such that $2^a7^b \equiv 2^{a'}7^{b'} \pmod{15}, 3^a5^b \equiv 3^{a'}5^{b'} \pmod{16}, a' \in \{0,1,2,3\}$ and $b' \in \{0,1\}$.
We conclude that both of the exponents can be changed by a number divisible by 4 without changing the remainder of dividing by the required number. Thus, we can consider only the case where $a, b \in \{0,1,2,3\}$. If $b \leq 1$, take $a = a'$ and $b' = b$; if $b > 1$, then $b' = b-2$ and a' can be chosen from the set $\{0,1,2,3\}$ so that it differs from the number a by exactly 2.
It remains to prove that the remainders of the numbers $2^{a'}7^{b'}$ when divided by 15 and the remainders of the numbers $3^{a'}5^{b'}$ when divided by 16 are pairwise different if the numbers a' and b'. come from the above mentioned sets. This can be seen by a manual check. ☐

Problem 1.14. If XYZ is a triangle with Y and Z being acute and different from $\frac{\pi}{4}$, and we also have L as the foot of the height from X, then prove that $\angle X$ is right if and only if

$$\frac{1}{XL-YL}+\frac{1}{XL-ZL}=\frac{1}{XL}$$

Solution. Let $XL=h, YL=x, ZL=y$. The stated equation then becomes $h^2=xy$ after some manipulation.
If $\angle X$ is right then by the right angle altitude theorem or geometric mean theorem we have $h^2=xy$.
On the other hand, suppose that $h^2=xy$. Clearly, $\cot\angle Y=\frac{x}{h}, \cot\angle Z=\frac{y}{h}$. Hence,

$$\cot(\angle Y+\angle Z)=\frac{\cot(\angle Y)\cot(\angle Z)-1}{\cot(\angle Y)+\cot(\angle Z)}$$

implying that $\angle Y+\angle Z=\frac{\pi}{2}$, and hence, $\angle X=\frac{\pi}{2}$. □

Problem 1.15. Given x and y as positive real numbers, where

$$\frac{x^3}{y^2}+\frac{y^3}{x^2}=5\sqrt{5xy}.$$

Show that

$$\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}=\sqrt{5}.$$

Solution. Let $a=\sqrt{\frac{x}{y}} and b=a+\frac{1}{a}$. Hence, we have that

$$\begin{aligned}
&\frac{x^3}{y^2}+\frac{y^3}{x^2}=5\sqrt{5xy}\\
\leftrightarrow & a^5+\frac{1}{a^5}=5\sqrt{5}\\
\leftrightarrow & \left(a+\frac{1}{a}\right)\left(a^4-a^2+1-\frac{1}{a^2}+\frac{1}{a^4}\right)=5\sqrt{5}\\
\leftrightarrow & b(b^4-5b^2+5)=5\sqrt{5}
\end{aligned}$$

$$\leftrightarrow (b-\sqrt{5})(b^4+\sqrt{5}b+5)$$

Since $b^4+\sqrt{5}b+5>0, b=\sqrt{5}$ and the rest follows. □

Problem 1.16. There are 20 cats priced from \$12 to \$15 and 20 sacks priced from 10 cents to \$1 for sale (all prices are different). Prove that each of two boys, John and Peter, can buy a cat in a sack by paying the same amount. of money.

Solution. The number of different possibilities for buying a cat and a sack is $20\cdot 20=400$ while the number of different possible prices is 1600 - 1210 + 1 = 391. Thus by the pigeonhole principle there exist two combinations of a cat and a sack costing the same amount of money. Note that the two cats (and also the two sacks) involved must be different as otherwise, the two sacks (respectively, cats) would have equal prices. □

Problem 1.17. Let p and q be two consecutive odd prime numbers. Prove that p + q is a product of at least two positive integers greater than 1 (not necessarily different).

Solution. Since $q-p=2k$ is even, we have $p+q=2(p+k)$. It is clear that $p<p+k<p+2k=q$. Therefore, $p+k$ is not prime and, consequently, is a product of two positive integers greater than 1. $\square$

Problem 1.18. There is a finite number of towns in a country. They are connected by one direction roads. It is known that, for any two towns, one of them can be reached from the other one. Prove that there is a town such that all the remaining towns can be reached from it.

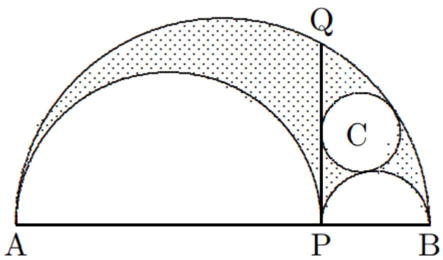
Solution. Consider a town A from which a maximal number of towns can be reached. Suppose there is a town B that cannot be reached from A. Then A can be reached from B and so one can reach more towns from B than from A, a contradiction. $\square$

Problem 1.19. Find all triples (x, y, z) of positive integers satisfying the system of equations:

$$\begin{cases} x^2 = 2(y+z) \\ x^6 = y^6 + z^6 + 31(y^2+z^2) \end{cases}$$

Solution. From the first equation, it follows that x is even. The second equation implies $x>y$ and $x>z$. Hence $4x > 2(y+z) = x^2$, and therefore $x=2$ and $y+z=2$, so $y=z=1$. It is easy to check that the triple $(2,1,1)$ satisfies the given system of equations. $\square$

Problem 1.20. In the figure below, you see three half-circles. The circle C is tangent to two of the half-circles and to the line $\overline{\text{PQ}}$ perpendicular to the diameter $\overline{\text{AB}}$. The area of the shaded region is 39π, and the area of the circle C is 9π. Find the length of the diameter $\overline{\text{AB}}$.



Solution. Let r and s be the radii of the half-circles with diameters AP and BP. Then we have

$$39\pi = \frac{\pi}{2}((r+s)^2 - r^2 - s^2) - 9\pi.$$

Hence, $rs = 48$. Let M be the midpoint of the diameter AB, N be the midpoint of PB, O be the centre of the circle C, and let F be the orthogonal projection of O on AB. Since the radius of C is 3, we have $|MO| = r+s-3, |MF| = r-s+3, |ON| = s+3$, and $|FN| = s-3$. Applying the Pythagorean theorem to the triangles MFO and NFO yields

$$(r+s-3)^2 - (r-s+3)^2 = |OF|^2 = (s+3)^2 - (s-3)^2$$

, which implies $r(s-3) = 3s$, so that $3(r+s) = rs = 48$. Hence, $|AB| = 2(r+s) = 32$ $\square$
